Ordered quantization and the Ehrenfest time scale

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We propose a prescription to quantize classical monomials in terms of symmetric and ordered expansions of noncommuting operators of a bosonic theory. As a direct application of such quantization rules, we quantize a classically time evolved function $\mathcal{O}(q,p,t)$, and calculate its expectation value in coherent states. The result can be expressed in terms of the application of a classical operator that performs a *Gaussian smoothing* of the original function $\mathcal{O}$ evaluated at the center of the coherent state. This scheme produces a natural semiclassical expansion for the quantum expectation values at a short time scale. Moreover, since the classical Liouville evolution of a Gaussian probability density gives the same form for the classical statistical mean value, we can calculate the first-order correction in $\hbar$ entirely from the associated classical time evolved function. This allows us to write a general expression for the Ehrenfest time in terms of the departure of the centroid of the quantum distribution from the classical trajectory, provided we start with an initially coherent state for each subsystem. In order to illustrate this approach, we have calculated analytically the Ehrenfest time of a model with N-coupled nonlinear oscillators with nonlinearity of even order.

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I. INTRODUCTION

Since the earliest days of the quantum theory, the investigation of the differences between the probabilistic concepts in classical Liouville and quantum dynamics has been an important issue. There have been many studies, in the last two decades, concerning the semiclassical regime of systems whose classical counterparts exhibit chaos [1]. The question of estimating how long the classical and quantum evolutions stay close has been one of the main questions in semiclassical analysis. For classically chaotic flows, the break time or Ehrenfest time (t_E) was found in Refs. [2,3] and then rigorously proved in Ref. [4] to diverge logarithmically with $\hbar$. In the classically regular flow, it was suggested in Ref. [5] that the behavior of t_E with $\hbar$ is algebraic ($\hbar^{-\delta}$), but no universal nature of such behavior has been shown yet.

According to the famous Ehrenfest's work [6], for quantum states that are localized enough, the time variation of the mean quantum momentum must be equal to the local force. This statement is exact for quadratic Hamiltonians, but its validity is restricted to a short-time scale, the Ehrenfest scale, for more general nonlinear systems. Mathematically, the Ehrenfest theorem allows us to write $\langle \mathcal{O}(\hat{X})\rangle = \mathcal{O}(\langle \hat{X}\rangle)$ for times smaller than the Ehrenfest time. In this situation, the initial dynamics is described essentially by a mean field approximation, where we have a localized packet obeying classical equations of motion. Then, it is reasonable to expect a rather good agreement between quantum and classical Liouvillian centroids and the classical trajectory. In fact, this scenario has already been reported in literature [7].

In this work, we propose a simple analytical scheme to calculate the Ehrenfest time for integrable systems. Our starting point is to propose a classical Liouvillian operator that makes explicit the symmetric form of the usual quantization rules. Using such an operator, we are able to take a general classical function $\mathcal{O}$ that expresses the time evolution of a physical quantity, quantize it, and evaluate its expectation value in coherent states at each time. The final result, which is shown to be analytically identical to the statistical average calculated through the classical Liouville formalism, is written in terms of a certain differential operator acting on the original classical function. Since this is just a compact form of expressing the action of the corresponding power series in $\hbar$, our recipe automatically leads to a semiclassical expansion around the time evolved classical function. Notice that the equality between quantum and statistical centroids guarantees that we are working within a classical time scale. This allows us to define mathematically the Ehrenfest scale using just the Liouvillian Gaussian average, without solving the quantum dynamical problem.

As an example, we present an explicit calculation of the Ehrenfest time for N-coupled nonlinear oscillators, with nonlinearity of order $2k$, an even integer. This model, to which we will address the quantum-classical departure issue can be associated to several nonlinearly interacting fields via Kerr-type [9,10] and cross-Kerr-type interactions [11] known to be relevant in quantum optics [12], and also for two quantized vibrational modes of a single trapped ion [13]. It is an integrable model where the role of nonlinearity can be studied analytically for several quantities and, in particular, for $N=k=2$ the quantum-classical *break time* has been determined [14] based on physical properties of the exact quantum states. Although the emphasis of studying the break times in the literature has been on "chaotic states" [8,15], here we are concerned with integrable cases, where we are able to derive an analytical expression for the Ehrenfest time.

II. ORDERED QUANTIZATION

A. Definition

Let us start by presenting a convenient quantization scheme for a single degree of freedom system, which will be easily generalized to degree N. Consider two noncommuting operators $\hat{A}$ and $\hat{B}$ whose commutator is a c-number, denoted by c ($[\hat{A},\hat{B}]=c$). Then, a given classical monomial $a^n b^m$, where a and b are canonically conjugated classical variables [16], will be quantized in a *symmetric* and *ordered* form through the prescriptions

$$a^n b^m \rightarrow \mathcal{S}_{\hat{A},\hat{B}}(\hat{A}^n\hat{B}^m), \tag{1}$$

where superoperator $\mathcal{S}_{\hat{A},\hat{B}}$ is given by

$$\mathcal{S}_{\hat{A},\hat{B}}=e^{-(1/2)[\hat{A},\hat{B}]\partial_{\hat{A}}\partial_{\hat{B}}}=\sum_{k=0}^{\infty}\frac{(-\frac{1}{2}[\hat{A},\hat{B}])^k}{k!}\partial_{\hat{A}}^k\partial_{\hat{B}}^k. \tag{2}$$

The above index $(\hat{A},\hat{B})$ corresponds to the ordering with $\hat{A}$ on the left and $\hat{B}$ on the right. Since $[\partial_{\hat{A}},\partial_{\hat{B}}]=[\partial_{\hat{A}},\hat{B}]=[\hat{A},\partial_{\hat{B}}]=0$, $\mathcal{S}_{\hat{A},\hat{B}}$ and $\mathcal{S}_{\hat{B},\hat{A}}$ are *classical* unitary differential superoperators satisfying $\mathcal{S}_{\hat{A},\hat{B}}\mathcal{S}_{\hat{B},\hat{A}}=\mathcal{S}_{\hat{B},\hat{A}}\mathcal{S}_{\hat{A},\hat{B}}=1$. The prescription defined in expression (1) leads to different orderings (depending on which variable is chosen to be a or b) for the same original classical function and, therefore, the associated symmetric operators must be the same. From this consideration, we deduce the following ordering formulas:

$$\hat{A}^n\hat{B}^m=\mathcal{S}^2_{\hat{B},\hat{A}}\hat{B}^m\hat{A}^n=e^{+[\hat{A},\hat{B}]\partial_{\hat{A}}\partial_{\hat{B}}}\hat{B}^m\hat{A}^n,$$

$$\hat{B}^m\hat{A}^n=\mathcal{S}^2_{\hat{A},\hat{B}}\hat{A}^n\hat{B}^m=e^{-[\hat{A},\hat{B}]\partial_{\hat{A}}\partial_{\hat{B}}}\hat{A}^n\hat{B}^m. \tag{3}$$

Using a classical displacement operator $e^{a\partial_x}f(x)=f(x+a)$, which can be used to write $e^{-c/2\partial_{\hat{A}}\partial_{\hat{B}}}\hat{A}^n\hat{B}^m=[\hat{A}-(c/2)\partial_{\hat{B}}]^n\hat{B}^m$, one can show that these results reproduce those in the textbooks (see, e.g., Louisell [17]). A simple example shows that our quantization scheme leads to the usual quantization rules, but in an automatically ordered form. Consider the following product of classical canonical phase space variables: q^2p. According to the usual rules, this is transformed into a *totally symmetric* operator [18] of the form $\frac{1}{3}(\hat{Q}^2\hat{P}+\hat{P}\hat{Q}^2+\hat{Q}\hat{P}\hat{Q})$, and using commutation relation $[\hat{Q},\hat{P}]=\imath\hbar$, rewritten as $\hat{Q}^2\hat{P}-\imath\hbar\hat{Q}$. But this is exactly the result produced by the expression in Eq. (1) with choice $(a,b)=(q,p)$, just by making some derivatives. We finally note that superoperators $\mathcal{S}_{\hat{A},\hat{B}}$ have already appeared in literature in the case of the canonical phase space operators $(\hat{A},\hat{B})=(\hat{Q},\hat{P})$, in slightly different contexts [19].

B. Normal ordering in bosonic operators

As an immediate application of the formulas presented above, we will express the totally symmetric ordered expression $\mathcal{S}_{\hat{Q},\hat{P}}\hat{Q}^n\hat{P}^m$ in a form more suitable for our purpose of taking expectation values in the Weyl-Heisenberg coherent states. We start by expressing such a polynomial in terms of bosonic creation and annihilation operators

$$\mathcal{S}_{\hat{Q},\hat{P}}\hat{Q}^n\hat{P}^m=e^{-(\imath\hbar/2)\partial_{\hat{Q}}\partial_{\hat{P}}}\hat{Q}^n\hat{P}^m$$
$$=z_{n,m}e^{(1/4)(\partial^2_{\hat{a}}-\partial^2_{\hat{a}^\dagger})}(\hat{a}+\hat{a}^\dagger)^n(\hat{a}-\hat{a}^\dagger)^m, \tag{4}$$

where $z_{nm}=(-\imath)^m(\sqrt{\hbar/2})^{n+m}$. Now, using

$$(\hat{a}\pm\hat{a}^\dagger)^n=\sum_{k=0}^{n}(\pm1)^{n-k}\binom{n}{k}e^{(1/2)\partial_{\hat{a}}\partial_{\hat{a}^\dagger}}(\hat{a}^{\dagger n-k}\hat{a}^k), \tag{5}$$

we can re-express Eq. (4) as

$$\mathcal{S}_{\hat{Q},\hat{P}}\hat{Q}^n\hat{P}^m=\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{m-l}z_{n,m}\binom{n}{k}$$
$$\times\binom{m}{l}e^{(1/4)(\partial^2_{\hat{a}}-\partial^2_{\hat{a}^\dagger})}e^{(1/2)\partial_{\hat{a}_2}\partial_{\hat{a}_1^\dagger}}e^{(1/2)\partial_{\hat{a}_4}\partial_{\hat{a}_3^\dagger}}$$
$$\times(\hat{a}_1^{\dagger n-k}\hat{a}_2^k\hat{a}_3^{\dagger m-l}\hat{a}_4^l), \tag{6}$$

with the subindexes that we introduced, to indicate where the action of the differentiation should take place. At the end of calculation we must erase all these subindexes. Using relation (3), we get

$$\mathcal{S}_{\hat{Q},\hat{P}}\hat{Q}^n\hat{P}^m=\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{m-l}z_{n,m}\binom{n}{k}$$
$$\times\binom{m}{l}e^{(1/4)(\partial^2_{\hat{a}}-\partial^2_{\hat{a}^\dagger})}e^{(1/2)\partial_{\hat{a}_2}\partial_{\hat{a}_1^\dagger}}e^{(1/2)\partial_{\hat{a}_4}\partial_{\hat{a}_3^\dagger}}e^{\partial_{\hat{a}_2}\partial_{\hat{a}_3^\dagger}}$$
$$\times(\hat{a}_1^{\dagger n-k}\hat{a}_3^{\dagger m-l}\hat{a}_2^l\hat{a}_4^k). \tag{7}$$

This is the normal- ordered expression (in the creation and annihilation operators) for the original $\hat{Q}\hat{P}$-ordered monomial. Now, it is a simple matter to calculate its expectation value.

C. Coherent states representation

Since we are interested in the connection between quantum and classical mechanics, the coherent state basis appears as the most appropriated one. In particular, it will be of interest for us to evaluate the expectation value in coherent states of some operator products like the one treated in preceding section. Then, we first calculate the matrix elements in the coherent state basis of the operator function given in Eq. (7)

$$\frac{\langle\alpha_1|\mathcal{S}_{\hat{Q},\hat{P}}\hat{Q}^m\hat{P}^m|\alpha_2\rangle}{\langle\alpha_1|\alpha_2\rangle}=z_{n,m}e^{(1/4)(\partial^2_{\alpha_2}-\partial^2_{\alpha_1^*})}\left\{e^{\partial_a\partial_d}e^{(1/2)\partial_a\partial_b}e^{(1/2)\partial_c\partial_d}\left[\sqrt{\frac{\hbar}{2}}(a+b)\right]^n\left[\sqrt{\frac{\hbar}{2}}\frac{(c-d)}{\imath}\right]^m\right\}_{\substack{a=c=\alpha_2\\ b=d=\alpha_1^*}}. \tag{8}$$

Setting now $\alpha_1=\alpha_2=\alpha_0=(q_0+\imath p_0)/\sqrt{2\hbar}$ and performing adequate variable transformations, we finally get

$$\langle\alpha_0|\mathcal{S}_{\hat{Q},\hat{P}}\hat{Q}^n\hat{P}^m|\alpha_0\rangle=e^{(\hbar/4)\nabla_0^2}q_0^n p_0^m, \tag{9}$$

where

$$\nabla_0^2=\partial_{q_0}^2+\partial_{p_0}^2,$$

$$(q_0,p_0)=(\langle\alpha_0|\hat{Q}|\alpha_0\rangle,\langle\alpha_0|\hat{P}|\alpha_0\rangle). \tag{10}$$

The expectation value given in Eq. (9) must be calculated through the series expansion of the classical operator $e^{(\hbar/4)\nabla_0^2}$, which gives a natural expansion in powers of $\hbar$, showing its semiclassical nature. In fact, in the classical limit $\hbar\rightarrow 0$, the quantum expectation value of the operator function reduces to a purely classical function calculated at the center of the coherent packet located at (q_0,p_0).

Another interesting relation can be obtained from a similar calculation:

$$\langle\hat{Q}^n\hat{P}^m\rangle=e^{(\hbar/4)\nabla_0^2}(e^{(\imath\hbar/2)\partial_{q_0}\partial_{p_0}}q_0^n p_0^m), \tag{11}$$

which implies that

$$\frac{1}{\imath\hbar}\langle[\hat{Q}^n,\hat{P}^m]\rangle=\exp\left[\frac{\hbar}{4}\nabla_0^2\right]\left(\frac{\sin(\hbar/2)\,\partial_{q_0}\partial_{p_0}}{\hbar/2}\right)q_0^n p_0^m. \tag{12}$$

It is important to note that the term within the parentheses in Eq. (11) is exactly the Weyl transform of operator $\hat{Q}^n\hat{P}^m$ [19]. The extra operator factor $e^{(\hbar/4)\nabla_0^2}$ is what makes reference to the width of the coherent packet, as we shall see later. These results also point to the existence of an asymptotic classical limit $(\hbar\rightarrow 0)$ for such expectation values in coherent states.

III. SHORT TIME QUANTIZATION

The usual quantization rules are defined in the Heisenberg picture, where the solutions for Hamilton's equations, $q_{\mathcal{H}}(q,p,t)$ and $p_{\mathcal{H}}(q,p,t)$, are transformed into Heisenberg operators $\hat{Q}_H(\hat{Q},\hat{P},t)$ and $\hat{P}_H(\hat{Q},\hat{P},t)$, where $\hat{Q}$ and $\hat{P}$ denote Schrödinger operators. On the other hand, since the Heisenberg and Schrödinger pictures coincide at $t=0$, any scheme of quantization based on the Schrödinger picture would describe reasonably the quantum world for very short times. However, as long as we are interested in analyzing the quantum operator evolution only during a classical (mean field) time scale, this would suffice (for the first order $\hbar$ correction see Ref. [3]).

In this context, consider a classically time evolved function $\mathcal{O}$ that has the following well defined expansion:

$$\mathcal{O}(q,p,t)=\sum_{n,m=0}^{\infty}c_{n,m}(t)q^n p^m. \tag{13}$$

Coefficients $c_{n,m}(t)$ contain all the time dependence. Now, applying the operator of ordered quantization $\mathcal{S}_{\hat{Q},\hat{P}}$ and taking the average in coherent states we obtain

$$\langle\alpha_0|\hat{\mathcal{O}}(\hat{Q},\hat{P},t)|\alpha_0\rangle=e^{(\hbar/4)\nabla_0^2}\mathcal{O}(q_0,p_0,t), \tag{14}$$

where we have used result (9). In order to estimate how long result (14) could be trusted, we present the calculation of the *classical statistical* counterpart of the problem. In the classical Liouvillian formalism, the following mean value is defined in the phase space (q,p):

$$\langle\mathcal{O}(q_0,p_0,t)\rangle_{cs}=\int dq\,dp\,\rho(q,p,t)\mathcal{O}(q,p,0), \tag{15}$$

where $(q_0,\ p_0)$ stands for the center of the initial distribution $\rho(q,p,0)$. Then, since $\rho(q,p,t)=e^{\mathcal{L}t}\rho(q,p,0)$ and $\mathcal{O}(q,p,t)=e^{-\mathcal{L}t}\mathcal{O}(q,p,0)$, taking into account the fact that the volume in the phase space is preserved, we can rewrite the previous equation in a Heisenberg-like form

$$\langle\mathcal{O}(q_0,p_0,t)\rangle_{cs}=\int dq\,dp\,\rho(q,p,0)\mathcal{O}(q,p,t). \tag{16}$$

Consider now a Gaussian initial distribution with width σ, and centered at point (q_0,p_0). Performing the variable transformations $(q-q_0)=x$ and $(p-p_0)=y$ and using the classical displacement operator again, we can rewrite Eq. (16) as

$$\langle\mathcal{O}\rangle_{cs}=\int dx\,dy\,\frac{e^{-x^2/\sigma}}{\sqrt{\pi\sigma}}\frac{e^{-y^2/\sigma}}{\sqrt{\pi\sigma}}e^{x\partial_q}e^{y\partial_p}\mathcal{O}(q,p,t). \tag{17}$$

Notice that function $\mathcal{O}$ inside the integral no longer depends on the integration variables. Hence, by completing squares and performing formally the integration, we finally obtain

$$\langle\mathcal{O}(q_0,p_0,t)\rangle_{cs}=e^{(\sigma/4)\nabla_0^2}\mathcal{O}(q_0,p_0,t). \tag{18}$$

Now, we have proved that the effect of applying operator $e^{(\sigma/4)\nabla_0^2}$ is exactly that of smoothing function $\mathcal{O}$ through a Gaussian mean, where σ is related to the width of the Gaussian distribution to be used in the smoothing. Accordingly, this operator has already been used to express the Q function as the Gaussian smoothing of the Wigner function [20]. It is important to notice that this result is exact for Gaussian statistical averages, i.e., it was derived from first principles without any approximation. The only implicit assumption used in the calculation was the existence of the derivatives in all orders for function $\mathcal{O}$.

The comparison between expressions (14) and (18), with $\sigma=\hbar$, confirms the fact that our proposal of a Schrödinger quantization for classical function should be adequate either at a short-time scale or in cases in which the classical function depends linearly on the phase space coordinates (q_0,p_0) (e.g., harmonic oscillator). Results (14) and (18) can easily be extended for higher degrees of freedom.

At this point one might formulate a question. First we interpret the action of operator $e^{(\hbar/4)\nabla_0^2}$ on any classical function as an exact factorization of the effect of a Gaussian wave packet contribution. In fact, for completely localized distribution ($\hbar=0$) there are no corrections. Then, one may ask: is it possible, in the particular case of coherent separable initial states, to express the exact quantum centroid, $\langle\hat{\mathbf{R}}(t)\rangle=\langle(\hat{Q}(t),\hat{P}(t))\rangle$ in terms of smoothed form $e^{(\hbar/4)\nabla^2}\mathbf{r_c}(t)$, where $\mathbf{r_c}(t)=(q_c(t),p_c(t))$ is a certain classical dynamical vector in phase space? The answer will be positive if we are able to calculate the inverse of the Gaussian smoothing operation on the quantum centroid vector in the phase space

$$\mathbf{r_c}(t)=e^{-(\hbar/4)\nabla_0^2}\langle\alpha_0|\hat{\mathbf{R}}(t)|\alpha_0\rangle. \tag{19}$$

Function $\mathbf{r_c}(t)$ will then be a *coherent quantum trajectory*, in the sense that it will carry all quantum dynamical information possible to put in a trajectory, except the contribution due to a Gaussian smoothing effect. For the trivial case of N noninteracting harmonic oscillators and the case of a bilinear coupling between two harmonic oscillators, the quantum-coherent and -classical trajectories coincide $((q_c(t),p_c(t))=(q_0(t),p_0(t)))$, as expected.

IV. THE EHRENFEST TIME

A. Formal definition

Now, we have all the necessary tools to undertake the problem of estimating the Ehrenfest time in the case of an initially separable coherent Gaussian wave packet. Assuming that the break instant occurs when the first-order correction in $\hbar$ becomes as important as the original vector, we apply the smoothing process to the phase space classical trajectory vector of a system with N degrees of freedom $\mathbf{r}(t)=(q_1(t),p_1(t),\ldots,q_N(t),p_N(t))$. Expanding the smoothing operator up to the first order in $\hbar$, we obtain

$$\langle\hat{\mathbf{R}}(t)\rangle=e^{(\hbar/4)\nabla^2}\mathbf{r}(t)\cong\mathbf{r}(t)+\frac{\hbar}{4}\nabla^2\mathbf{r}(t), \tag{20}$$

where $\nabla^2=\Sigma_{i=1}^N\nabla_i^2$ is the N-dimensional Laplacian operator. Now, we formally define Ehrenfest time t_E as being the instant at which the magnitude of the difference between the quantum centroid and the corresponding classical vector in phase space becomes equal to the magnitude of the initial classical vector. Mathematically this condition can be expressed as follows:

$$\frac{||\langle\hat{\mathbf{R}}(t_E)\rangle-\mathbf{r}(t_E)||}{||\mathbf{r}(0)||}=1. \tag{21}$$

This is similar to the definition for t_E given in Ref. [3]. Now, using Eq. (20), we obtain the prescription for the analytical calculation of the Ehrenfest time:

$$\frac{\hbar}{4}\frac{||\nabla^2\mathbf{r}(t_E)||}{||\mathbf{r}(0)||}=1. \tag{22}$$

It is remarkable that we do *not* need to solve the quantum equations of motion to find t_E. This is the main difference with the approach used in Refs. [3,22]. In fact, our scheme takes into account only the wave packet correction to the classical dynamics. However, since that is indeed the first manifestation of short-time quantum effects [6], our proposal must be enough to give a good estimate for t_E. In what follows, we will calculate the above defined break time for the system of N-coupled nonlinear oscillators.

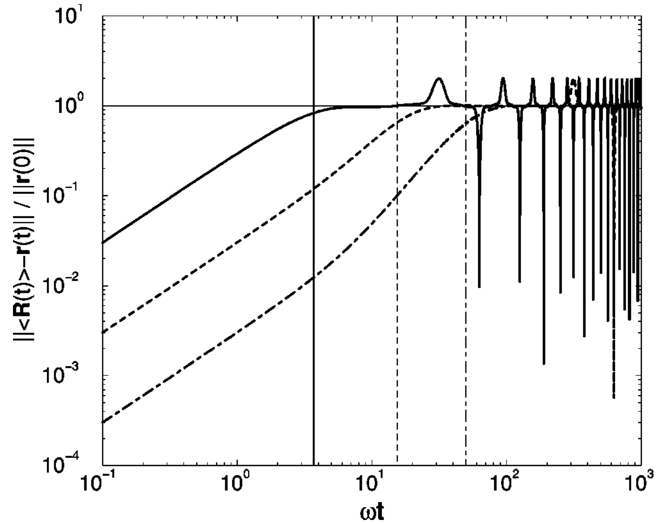

FIG. 1. Dimensionless quantity which measures the departure between quantum and classical centroids as a function of the dimensionless parameter ωt for the bidimensional quartic oscillator ($N=2,k=2$). These calculations were analytically performed with $q_i=p_i=1$ ($\Lambda=2$), $\omega_i=\omega=1$, and $g=0.1$, and $\hbar=1$ (solid line), $\hbar=0.1$ (dashed line) and $\hbar=0.01$ (dot-dashed line). The Ehrenfest time scale (25) for each value of $\hbar$ is represented by a vertical line in the corresponding style.

B. Application to a nonlinear system

Consider the following classical Hamiltonian:

$$\mathcal{H}=\sum_{i=1}^N\omega_i\left(\frac{p_i^2+q_i^2}{2}\right)+g\left[\sum_{i=1}^N\left(\frac{p_i^2+q_i^2}{2}\right)\right]^k, \tag{23}$$

where $k\geq1$ is integer and g is the only coupling constant of the system, and from which we define the characteristic classical action $\Lambda=\Sigma_{i=1}^N[(p_i^2+q_i^2)/2]$ [21]. The equations of motion can be solved by noticing that Λ itself is a constant of motion. The result reads

$$\begin{bmatrix}q_i(t)\\p_i(t)\end{bmatrix}=\begin{bmatrix}\cos\Theta_i(t) & \sin\Theta_i(t)\\-\sin\Theta_i(t) & \cos\Theta_i(t)\end{bmatrix}\begin{bmatrix}q_i\\p_i\end{bmatrix}, \tag{24}$$

where (q_i,p_i) are the initial conditions of the ith oscillator and $\Theta_i(t)=\omega_i t+gtk\Lambda^{k-1}$ is a rotation angle in phase space. Finally, using Eqs. (22) and (24), we get for the Ehrenfest time of this system

$$t_E\cong\left[\frac{1}{k(k-1)g\Lambda^{k-1}}\left(\frac{2\Lambda}{\hbar}\right)^{0.5}\right]\left(1-\frac{\hbar k^2}{8\Lambda}\right), \tag{25}$$

where we kept only the first-order terms in $\hbar$. For case ($N=1,k=2$), our estimate gives the same scale as the exact calculation given in Ref. [22], $t_E \cong (1/\mu)\sqrt{I_o/\hbar}$ (identifying Λ and g with I_o and μ, respectively). Also, case $(N=2,k=2)$ reproduces the results in Ref. [23]. Trivial limits are also contemplated by the above result, namely, the cases of a harmonic system ($g=0$, $k=0$ or $k=1$), for which $t_E \to \infty$. Moreover, our result is in accordance with some conjectures predicting the general form $(1/\Omega)(S/\hbar)^{\delta}$ for the break time of classically integrable systems [3,24], where $\Omega = 1/[k(k-1)g\Lambda^{k-1}]$ and $S=2\Lambda$ are, respectively, the typical frequency and classical action of the system in consideration. We illustrate in Fig. 1 the Ehrenfest scale predicted by expression (25) for case $(N=2,k=2)$, where we can see an algebraic ($\sim t^2$) short-time departure. We also note that the first-order correction to the Ehrenfest time scale seems to indicate a more appropriated parameter to measure the classicality in this system: $\hbar k^2/8\Lambda \ll 1$.

V. CONCLUSION

In this paper we proposed a scheme for an ordered symmetric quantization based on the action of certain differential operators. As an application, we quantized a classically time evolved dynamical functions of the canonical variables in phase space, and showed that such a procedure is adequate during the Ehrenfest time scale for separable coherent initial states. This allowed us to propose a formal definition for the Ehrenfest time in terms of the phase space Laplacian operator acting on the classical solutions of the equations of motion. This makes our calculation much simpler than other approaches. We performed an explicit calculation for a system of N-coupled nonlinear ($2k$th order) oscillators and calculated the Ehrenfest time for general $N \geqslant 1$ and $k \geqslant 1$. The results are shown to agree with the results known in the literature for some particular cases.

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[1] M.B. Berry, Proc. R. Soc. London, Ser. A **413**, 183 (1987); M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990); *Chaos and Quantum Physics*, Les Houches Lectures LII, edited by M.-J. Giannoni, A. Voros and J. Zinn-Justin (North-Holland, Amsterdam, 1991).

[2] G.P. Berman and G.M. Zaslavsky, Physica A **91**, 450 (1978); G.M. Zaslavsky, Phys. Rep. **80**, 157 (1981).

[3] G.P. Berman, E.N. Bulgakov, and D.D. Holm, *Crossover-Time in Quantum Boson and Spin Systems*, Lecture Notes in Physics (Springer-Verlag, Heidelberg, 1994).

[4] G.A. Hagerdorn and A. Joye, Ann. Inst. Henri Poincare, Sect. A **1**, 837 (2000); M. Combescure and D. Robert, Asymptotic Anal. **14**, 377 (1997); D. Bambusi *et al.*, *ibid.* **21**, 149 (1999).

[5] Y.-C. Lai, E. Ott, and C. Greborgi, Phys. Lett. A **173**, 148 (1993).

[6] P. Ehrenfest, Z. Phys. **45**, 455 (1927).

[7] L.E. Ballentine and S.M. McRae, Phys. Rev. A **58**, 1799 (1998).

[8] L.E. Ballentine, Phys. Rev. A **65**, 062110 (2002).

[9] G.J. Milburn, Phys. Rev. A **33**, 674 (1986); G.J. Milburn and C.A. Holmes, Phys. Rev. Lett. **56**, 2237 (1986).

[10] M. Kitagawa and Y. Yamamoto, Phys. Rev. A **34**, 3974 (1986).

[11] V. Peřinová and A. Lukš, in *Progress in Optics XXXIII*, edited by E. Wolf (North-Holland, Amsterdam, 1994).

[12] M.O. Scully and M.S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).

[13] J. Steinbach, J. Twamley, and P.L. Knight, Phys. Rev. A **56**, 4815 (1997), and references therein.

[14] L. Sanz, R.M. Angelo, and K. Furuya, e-print quant-ph/0210162.

[15] Z.P. Karkuszewski, J. Zakrzewski, and W.H. Zurek, Phys. Rev. A **65**, 042113 (2002).

[16] We are considering the cases where a and b are associated to the operators $\hat{A}$ and $\hat{B}$ through the coherent state mean values, e.g., $\langle \alpha_o | \hat{A} | \alpha_o \rangle$. The canonical pairs $(\hat{a},\hat{a}^{\dagger})$ and $(\hat{Q},\hat{P})$ are both contemplated in our notation.

[17] W. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964).

[18] K.E. Cahill and R.J. Glauber, Phys. Rev. **177**, 1857 (1969).

[19] A. Royer, Phys. Rev. A **43**, 44 (1991); M.A. Marchiolli, Rev. Bras. Ens. Física (to be published); D. Galetti, notes on the *III Escola Mário Schenberg*, DF-UFPB, João Pessoa (1996).

[20] Héctor Moya-Cessa and Peter L. Knight, Phys. Rev. A **48**, 2479 (1993).

[21] Notice that here we have an extra factor of $\frac{1}{2}$ as compared to Ref. [14].

[22] G.P. Berman, A.M. Iomin, and G.M. Zaslavsky, J. Phys. D **4**, 113 (1981).

[23] A.C. Oliveira, K.F. Romero, and M.C. Nemes, Phys. Rev. E (to be published).

[24] W.H. Zurek and J.P. Paz, e-print quant-ph/9612037; W.H. Zurek and J.P. Paz, in *Proceedings of the Fourth Drexel Meeting*, edited by D.H. Feng *et al.* (Plenum, New York, 1995).